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RELATION OF A FREQUENCY-DEPENDENT STRUCTURE WITH THE CORRESPONDING FREQUENCY-INDEPENDENT STRUCTURE

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Abstract—It is shown that the second theorem in the previous paper [Nakamura and Takewaki (1989) Optimal elastic structures with frequency-dependent elastic supports. *International Journal of Solids and Structures* **25**(5), 539–551] can be extended to a more general case where there is no special restriction on the characteristics of frequency-dependent stiffnesses except their positive definiteness and a characteristic as a single-valued function. This theorem enables one to disclose the relationship between the design space of elastic structures with frequency-dependent stiffnesses and that of elastic structures with the corresponding frequency-independent stiffnesses, both with respect to fundamental natural frequency. It facilitates not only clarification of lowest-mode qualification conditions for the frequency-dependent model with the help of the frequency-independent model, but also use of the modal analysis technique via the substitute model (the frequency-independent model). © 1998 Elsevier Science Ltd.

1. INTRODUCTION

Fundamental features of an elastic structure including members with frequency-dependent stiffnesses have been disclosed for the first time in the previous paper (Nakamura and Takewaki, 1989). Two new theorems have been introduced and proved in the case where all the frequency-dependent stiffnesses are expressed as single-valued non-increasing positive functions of frequency. Those two theorems have been utilized for establishing one-to-one correspondence between the design spaces of an ordered set of elastic frames supported by members with frequency-dependent stiffnesses and of the corresponding ordered set of elastic frames supported by those with the corresponding frequency-independent stiffnesses, both with respect to fundamental natural frequency. The optimal solution to a problem of optimum design of the former frames subject to an equality constraint on fundamental natural frequency has been shown to coincide with that of the latter frames (Nakamura and Takewaki, 1989). It has also been shown there that the optimal solution to the problem of optimum design of the former frames for specified fundamental natural frequency is also the optimal solution to the problem subject to the corresponding inequality constraint on fundamental natural frequency under the same restriction on supporting members.

It is noticeable that, while an iterative algorithm, e.g. the determinant-search procedure, is necessary in finding eigenvalues of an elastic structure including members with frequency-dependent stiffnesses, no such iterative algorithm is required in a hybrid inverse eigenmode problem (Takewaki and Nakamura, 1995, 1997; Takewaki, 1996; Takewaki *et al.*, 1996). This merit results from the fact that the stiffness of a frequency-dependent member is determined directly by the specification of a fundamental natural frequency in the context of the inverse problem. This advantage facilitates the formulation of a hybrid inverse eigenmode problem for frequency-dependent structures. After the determination of stiffnesses of the frequency-dependent members, a theory (Takewaki and Nakamura, 1997) of hybrid inverse eigenmode problems for the frequency-independent model can be applied to the frequency-dependent model. However, it is absolutely necessary in such a case to disclose the relation between the design space of the frequency-dependent model and that of the corresponding frequency-independent model with respect to fundamental natural

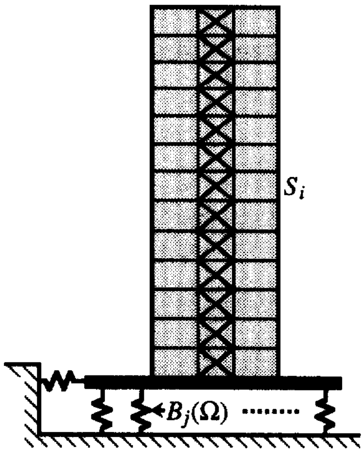

Fig. 1. Elastic structure consisting of elastic members or elements with frequency-independent stiffnesses $\mathbf{S} = \{S_i\}$ and those with frequency-dependent stiffnesses $\mathbf{B}(\Omega) = \{B_j(\Omega)\}$.

frequency. It should also be pointed out here that Gladwell's pioneering work (Gladwell, 1986) in the field of fully inverse eigenmode problems exists.

The purpose of this paper is to show that the second theorem in the previous paper (Nakamura and Takewaki, 1989) can be extended to a more general case where there is no special restriction on the characteristics of frequency-dependent stiffnesses except their positive definiteness and a characteristic as a single-valued function. This new theorem discloses that the design space of elastic structures including elastic members with frequency-dependent stiffnesses with respect to fundamental natural frequency is included within that of elastic structures with the corresponding frequency-independent stiffnesses. It is further shown that the new theorem facilitates not only clarification of lowest-mode qualification conditions for the frequency-dependent model with the help of the frequency-independent model, but also use of the modal analysis technique via the substitute model (the frequency-independent model).

2. A NEW THEOREM ON FREQUENCY-DEPENDENT AND FREQUENCY-INDEPENDENT STRUCTURES

Consider an elastic structure, as shown in Fig. 1, consisting of elastic members or elements with frequency-independent stiffnesses $\mathbf{S} = \{S_i\}$ and those with frequency-dependent stiffnesses $\mathbf{B}(\Omega) = \{B_j(\Omega)\}$. It is assumed that $\mathbf{S}$ and $\mathbf{B}(\Omega)$ are given. This elastic structure is assumed to have N degrees-of-freedom in the dynamic response and to have the total mass matrix $\mathbf{M}$ of $N \times N$. This matrix $\mathbf{M}$ may be a combination of a consistent mass matrix and a lumped mass matrix. Let $\mathbf{K}_{\mathrm{FD}}(\mathbf{S}, \mathbf{B}(\Omega))$ denote the stiffness matrix of $N \times N$ of the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$. For the sake of simplicity of expression, the elastic structure consisting of $\mathbf{S}$ and $\mathbf{B}(\Omega)$ is referred to as "the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$". Let $\Omega_1(=\omega_1^2)$ denote the lowest eigenvalue of the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$, i.e. square of the fundamental natural circular frequency ω_1. In order to determine Ω_1, a nonlinear eigenvalue problem has to be solved via a numerical procedure, e.g. the Wittrick and Williams (1971) algorithm or the determinant-search procedure.

The governing equations of eigenvibration of order s and t of the nonlinear eigenvalue problem for the structure $\{\mathbf{S}, \mathbf{B}(\Omega)\}$ may be described as follows:

$$[\mathbf{K}_{\mathrm{FD}}(\mathbf{S}, \mathbf{B}(\Omega_s)) - \Omega_s \mathbf{M}]\mathbf{U}_{\mathrm{FD}}^{(s)} = \mathbf{0} \tag{1a}$$

$$[\mathbf{K}_{\mathrm{FD}}(\mathbf{S}, \mathbf{B}(\Omega_t)) - \Omega_t \mathbf{M}]\mathbf{U}_{\mathrm{FD}}^{(t)} = \mathbf{0} \tag{1b}$$

where Ω_s and Ω_t are the sth and tth eigenvalues of the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$ and $\mathbf{U}_{\mathrm{FD}}^{(s)}$ and $\mathbf{U}_{\mathrm{FD}}^{(t)}$ are the sth and tth eigenvectors of the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$. While premultiplication of transposes of $\mathbf{U}_{\mathrm{FD}}^{(l)}$ and $\mathbf{U}_{\mathrm{FD}}^{(s)}$ in eqns (1a, b) and subtraction of both sides lead to orthogonality conditions of the eigenvectors in the frequency-independent case, those conditions can not be derived in this case, i.e.

$$\mathbf{U}_{\mathrm{FD}}^{(l)\,\mathrm{T}}\mathbf{K}_{\mathrm{FD}}(\mathbf{S},\mathbf{B}(\Omega_s))\mathbf{U}_{\mathrm{FD}}^{(s)}-\mathbf{U}_{\mathrm{FD}}^{(s)\,\mathrm{T}}\mathbf{K}_{\mathrm{FD}}(\mathbf{S},\mathbf{B}(\Omega_l))\mathbf{U}_{FD}^{(l)}\neq 0. \tag{2}$$

Since modal analysis can reduce significantly computational efforts especially in a structure with a large number of degrees-of-freedom, an approximate frequency-independent model (substitute model) is introduced here, i.e. an elastic structure consisting of $\mathbf{S}$ and elastic members with frequency-independent stiffnesses $\bar{\mathbf{B}} = \mathbf{B}(\Omega_1) = \{B_j(\Omega_1)\}$. This structure is referred to as "the structure of $\{\mathbf{S},\bar{\mathbf{B}}\}$". The stiffness matrix $\mathbf{K}_{\mathrm{FI}}(\mathbf{S},\bar{\mathbf{B}})$ of the structure of $\{\mathbf{S},\bar{\mathbf{B}}\}$ is derived by replacing $\mathbf{B}(\Omega)$ by $\bar{\mathbf{B}}$ in the matrix $\mathbf{K}_{\mathrm{FD}}(\mathbf{S},\mathbf{B}(\Omega))$. It should be kept in mind that, if the frequency-independent model has the same fundamental natural frequency and lowest eigenmode as those of the frequency-dependent model, the response of the latter under a dynamic disturbance with wide-band frequency characteristics can be approximated by that of the former within a reasonable accuracy in most of cantilever-type mechanical and civil engineering structures [for example, see Tsai (1974); Bielak (1976); Takewaki (1991)].

In this paper, it is assumed that each frequency-dependent stiffness of $\mathbf{B}(\Omega)$ is a single-valued positive function of Ω. Therefore, its static stiffness is positive, i.e.

$$B_j(0) > 0 \quad (\text{for all } j). \tag{3}$$

Then it is apparent that the structure of $\{\mathbf{S},\{B_j(0)\}\}$ has a positive fundamental natural frequency. In this case, the following theorem holds. It should be noted that the following theorem is a generalization of the second theorem in the previous paper (Nakamura and Takewaki, 1989) (a property of $B_j(\Omega)$ as a non-increasing function has been removed).

2.1. *Theorem A*

Let $\omega_1(=\sqrt{\Omega_1})$ denote the fundamental natural circular frequency of the structure of $\{\mathbf{S},\mathbf{B}(\Omega)\}$ where every stiffness in $\mathbf{B}(\Omega)$ is a single-valued positive function of Ω. Then the structure of $\{\mathbf{S},\bar{\mathbf{B}}(=\mathbf{B}(\Omega_1))\}$ has the same set of the fundamental natural circular frequency and the fundamental eigenvector as that of the former structure.

2.2. *Proof*

Let $\mathbf{U}_{\mathrm{FD}}^{(1)}$ and $\mathbf{U}_{\mathrm{FI}}$ denote the lowest eigenvector of the structure of $\{\mathbf{S},\mathbf{B}(\Omega)\}$ and an eigenvector of the structure of $\{\mathbf{S},\bar{\mathbf{B}}\}$. Since the structure of $\{\mathbf{S},\mathbf{B}(\Omega)\}$ has ω_1 as the fundamental natural frequency, it is evident that the structure of $\{\mathbf{S},\bar{\mathbf{B}}\}$ has ω_1 as *one* of the natural frequencies. This fact may be described as follows:

$$[\mathbf{K}_{\mathrm{FD}}(\mathbf{S},\mathbf{B}(\Omega_1))-\Omega_1\mathbf{M}]\mathbf{U}_{\mathrm{FD}}^{(1)} = [\mathbf{K}_{\mathrm{FI}}(\mathbf{S},\bar{\mathbf{B}})-\Omega_1\mathbf{M}]\mathbf{U}_{\mathrm{FI}} = \mathbf{0}. \tag{4}$$

Theorem A may be proved by showing that the structure of $\{\mathbf{S},\bar{\mathbf{B}}\}$ has ω_1 as one of the natural frequencies, but will not have any other natural frequencies smaller than ω_1.

Let us define a new structure of $\{\mathbf{S},\mathbf{B}(\lambda)\}$ where λ is a *specified* positive value. The structure of $\{\mathbf{S},\mathbf{B}(\lambda)\}$ with a specified value λ is different from the structure of $\{\mathbf{S},\mathbf{B}(\Omega)\}$ and is a structure consisting of only elastic members with frequency-independent stiffnesses. The governing equation of eigenvibration of the structure of $\{\mathbf{S},\mathbf{B}(\lambda)\}$ may be expressed as [see Lancaster (1966)]

$$[\mathbf{K}_{\mathrm{FI}}(\mathbf{S},\mathbf{B}(\lambda))-\Omega_k^*\mathbf{M}]\mathbf{U}_{\mathrm{FI}}^{*\,(k)} = \mathbf{0} \tag{5}$$

In eqn (5), Ω_k^* and $\mathbf{U}_{\mathrm{FI}}^{*\,(k)}$ denote the kth eigenvalue and the kth eigenvector of the structure of $\{\mathbf{S},\mathbf{B}(\lambda)\}$ with a specified value λ. Since Ω_k^* is the kth eigenvalue calculated for a specified

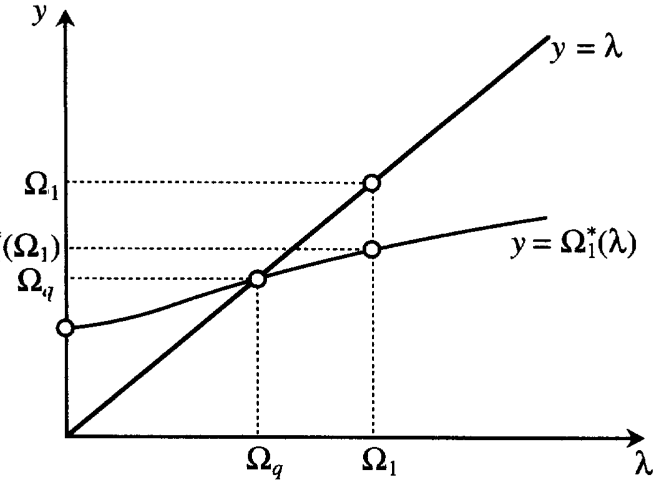

Fig. 2. Relation between the fundamental natural frequency of a frequency-dependent structure and that of a frequency-independent structure.

value λ and can be regarded as a function of λ through eqn (5), it will be denoted by $\Omega_k^*(\lambda)$ in the sequel. If the numbering of the eigenvalues is implemented from the positive minimum value according to the magnitude, $\Omega_k^*(\lambda)$ can be regarded as a single-valued positive function of λ.

On the other hand, the eigenvalue problem for the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$ may be expressed as

$$[\mathbf{K}_{\mathrm{FD}}(\mathbf{S}, \mathbf{B}(\Omega_k)) - \Omega_k \mathbf{M}]\mathbf{U}_{\mathrm{FD}}^{(k)} = \mathbf{0}. \tag{6}$$

It is apparent from eqns (5) and (6) that the value λ satisfying the equation $\Omega_k^*(\lambda) = \lambda$ is one of the eigenvalues of the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$. In other words, every eigenvalue of the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$ can be characterized by the value λ at the intersection of the function, $y = \Omega_k^*(\lambda)$, and the straight line, $y = \lambda$.

Now assume that the structure of $\{\mathbf{S}, \bar{\mathbf{B}}\}$ has ω_1 as one of the natural frequencies other than the fundamental natural frequency. If the fundamental natural frequency of the structure of $\{\mathbf{S}, \bar{\mathbf{B}}\}$ is denoted by $\tilde{\omega}_1(\tilde{\omega}_1^2 = \Omega_1^*(\Omega_1))$, the function $y = \Omega_1^*(\lambda)$ necessarily intersects with the straight line $y = \lambda$ at a point Ω_q smaller than Ω_1 because $\tilde{\omega}_1 < \omega_1$ and the function $y = \Omega_1^*(\lambda)$ is a single-valued positive function. This fact is shown in Fig. 2. The value Ω_q at the intersection indicates one of the eigenvalues of the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$. Therefore, it is drawn that the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$ has a natural frequency smaller than ω_1. This consequence apparently contradicts the assumption that the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$ has the fundamental natural frequency ω_1. It is, therefore, concluded that the structure of $\{\mathbf{S}, \bar{\mathbf{B}}\}$ has ω_1 as the fundamental natural frequency.

Since the structure of $\{\mathbf{S}, \mathbf{B}(\Omega)\}$ and the structure of $\{\mathbf{S}, \bar{\mathbf{B}}\}$ have the same stiffness and mass matrices due to $\mathbf{K}_{\mathrm{FD}}(\mathbf{S}, \mathbf{B}(\Omega_1)) = \mathbf{K}_{\mathrm{FI}}(\mathbf{S}, \bar{\mathbf{B}})$, it is evident that they have the same fundamental eigenvector. This completes the proof. ■

It should be noted that the fact explained in Fig. 2 can be shown to be also valid for the case where the structure of $\{\mathbf{S}, \mathbf{B}(\lambda)\}$, e.g. a two-bar truss shown in Fig. 3, has multiple eigenvalues smaller than Ω_1. It is noted that $B_1(\Omega) = k_1(\Omega)$ and $S_1 = k_2$ in this example. The frequency-dependent stiffness $k_1(\Omega)$ and the frequency-independent stiffness k_2 indicate the stiffness between an axial force and an axial elongation. An example of these stiffnesses is shown in Fig. 4(a). The relation corresponding to Fig. 2 is shown in Fig. 4(b). At the point of multiple eigenvalues both bars of the structure of $\{\mathbf{S}, \mathbf{B}(\lambda)\}$ have the same stiffness for a special λ value and a purely horizontal mode and a purely vertical mode become the corresponding eigenmodes. It can be observed from Fig. 4(b) that it is sufficient to adopt the same numbering procedure, $\Omega_1^*(\lambda), \Omega_2^*(\lambda), \ldots$, of the eigenvalues as stated before, i.e.

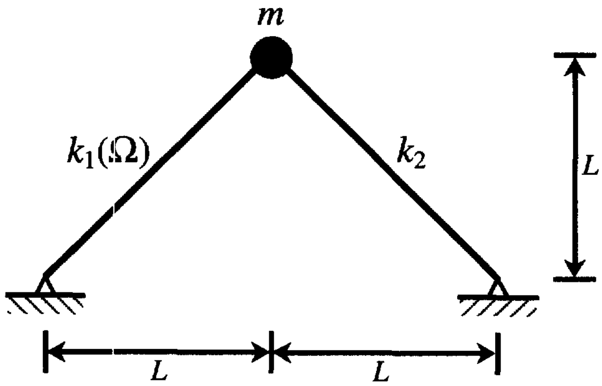

Fig. 3. Two-bar truss consisting of a bar with a frequency-dependent stiffness and one with a frequency-independent stiffness.

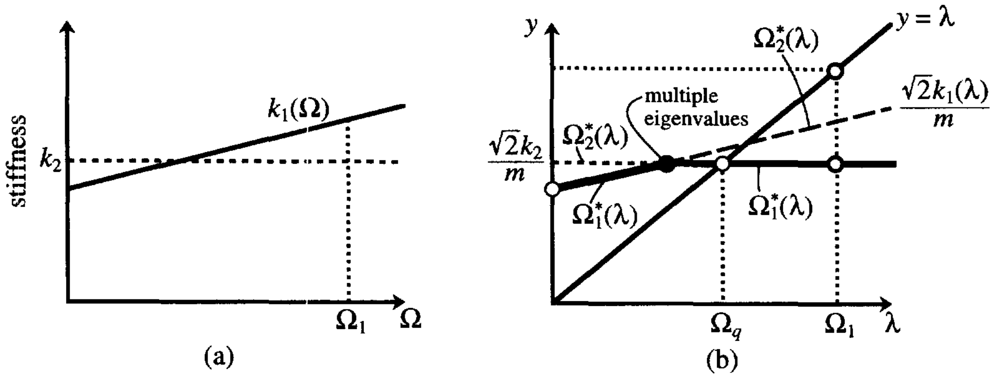

Fig. 4. (a) Frequency-dependent stiffness $k_1(\Omega)$ with respect to Ω and frequency-independent stiffness k_2; (b) relation between the fundamental natural frequency of a frequency-dependent structure and that of a frequency-independent structure in the case where multiple eigenvalues exist in the range smaller than Ω_1.

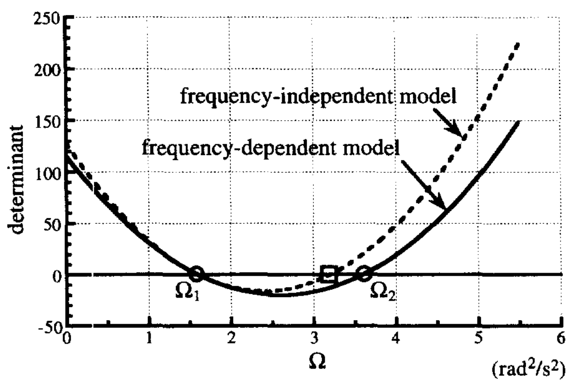

Fig. 5. Plot of det $[\mathbf{K}_{FD}(\mathbf{S}, \mathbf{B}(\Omega)) - \Omega\mathbf{M}]$ and det $[\mathbf{K}_{FI}(\mathbf{S}, \bar{\mathbf{B}}) - \Omega\mathbf{M}]$ with respect to Ω.

numbering the order of eigenvalues from the smallest. As in Fig. 2, $y = \Omega_1^*(\lambda)$ necessarily intersects with $y = \lambda$ at Ω_q smaller than Ω_1.

No restriction has been introduced on the characteristics of $\mathbf{B}(\Omega)$ except their positive definiteness and a characteristic as a single-valued function in the proof. Therefore, $B_j(\Omega)$ may be any single-valued positive function of Ω.

Figure 5 shows the plots of det $[\mathbf{K}_{FD}(\mathbf{S}, \mathbf{B}(\Omega)) - \Omega\mathbf{M}]$ and det $[\mathbf{K}_{FI}(\mathbf{S}, \bar{\mathbf{B}}) - \Omega\mathbf{M}]$ with respect to Ω for the two-bar truss shown in Fig. 3. In this example, the following numerical data have been used; $k_1(\Omega) = c\,\Omega + d$, $c = 1.0$ (N·s²/m), $d = 14.4$ (N/m), $k_2 = 8.0$ (N/m), $m = 5.0$ (kg). The lowest eigenvalue of the frequency-dependent model has been found to

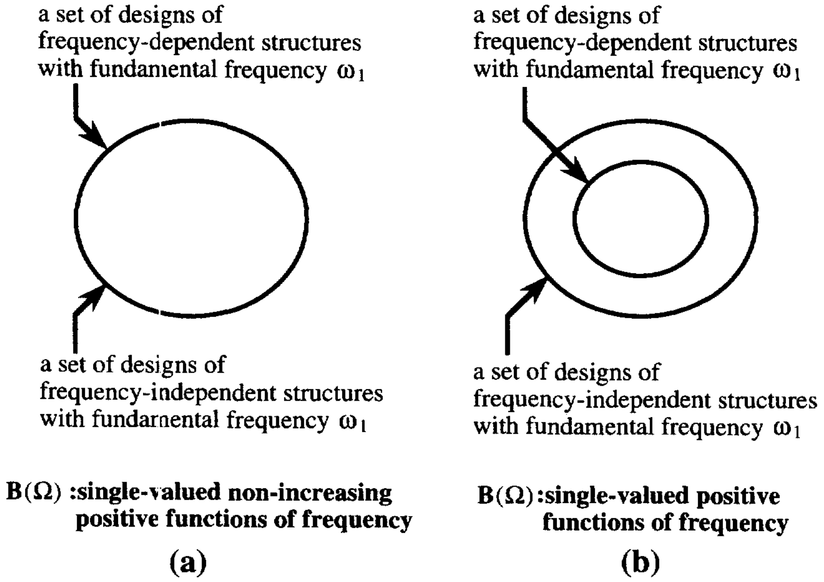

Fig. 6. Correspondence of design spaces of frequency-dependent structures and of frequency-independent structures: (a) $\mathbf{B}(\Omega)$: single-valued non-increasing positive functions of frequency; (b) $\mathbf{B}(\Omega)$: single-valued positive functions of frequency.

be $\Omega_1 = 1.6$ (rad^2/s^2) and $\bar{\mathbf{B}}$ has been determined based on $\Omega_1 = 1.6$ (rad^2/s^2). Figure 5 indicates that the frequency-dependent model and the corresponding frequency-independent model have the same lowest eigenvalue $\Omega_1 = 1.6$ (rad^2/s^2) and demonstrates validity of Theorem A.

3. CORRESPONDENCE OF DESIGN SPACES OF TWO CLASSES OF STRUCTURES

In the previous paper (Nakamura and Takewaki, 1989), it has been shown that there exists one-to-one correspondence between the design spaces of an ordered set of elastic frames supported by members with frequency-dependent stiffnesses and of the corresponding ordered set of elastic frames supported by those with the corresponding frequency-independent stiffnesses, both with respect to fundamental natural frequency, so long as the former stiffnesses are single-valued non-increasing positive functions of frequency [see Fig. 6(a)].

Theorem A in this paper implies that the design spaces of these two classes of structures have the relationship as shown in Fig. 6(b) in the case where there is no special restriction on the characteristics of $\mathbf{B}(\Omega)$ except their positive definiteness and a characteristic as a single-valued function. The fact that there does not exist a one-to-one correspondence between these two classes of design spaces is evident from the first example in the previous paper (Nakamura and Takewaki, 1989).

Since qualification conditions on the lowest eigenmode are derivable for a certain class of structures (Takewaki and Nakamura, 1995), the one-to-one correspondence between two kinds of structures is expected to lead to clarification of those conditions for a frequency-dependent structure. This subject will be discussed elsewhere.

4. CONCLUSIONS

It has been shown that the second theorem introduced in the previous paper (Nakamura and Takewaki, 1989) can be extended to the case where there is no special restriction on the characteristics of frequency-dependent stiffnesses except their positive definiteness and a characteristic as a single-valued function. This new theorem implies that the design space of elastic structures including elastic members with frequency-dependent stiffnesses with respect to fundamental natural frequency is included in that of elastic structures including elastic members with the corresponding frequency-independent stiffnesses. It is expected that this theorem facilitates clarification of lowest-mode qualification conditions for the frequency-dependent model with the help of the corresponding frequency-independent model.

It is also expected that, while the response analysis of a frequency-dependent model under external loading requires an analysis in the frequency domain and any proposed modal analysis technique cannot be used, the frequency-independent model (substitute model) enables one to utilize the well-known modal analysis technique and would enhance computational efficiency within a reasonable accuracy.

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